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Combinatorics of the Dimer Model on a Strip

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ABSTRACT

In this note, we give a closed formula for the partition function of the dimer model living on a $2 \times n$ strip of squares or hexagons on the torus for arbitrary even n. The result is derived in two ways, by using a Potts model like description for the dimers, and via a recursion relation that was obtained from a map to a $1D$ monomer–dimer system.

The problem of finding the number of perfect matchings can also be translated to the problem of finding a minmal feedback arc set on the dual graph.

1 Introduction

In this note, we give a closed formula for the partition function of the dimer model living on a $2 \times n$ strip of squares or hexagons on the torus for arbitrary even n.

The dimer model is concerned with the statistical mechanics of close packed dimer arrangements on a bipartite graph. The real–world representation of the dimer model is the adsorption of diatomic molecules on a crystal surface.

In the 1960s, the question of how many perfect matchings exist on a plane graph was solved independently by Kasteleyn [1, 2], and Temperley and Fisher [3, 4]: the total number is given by the Pfaffian of a signed, weighted adjacency matrix of the graph (the Kasteleyn matrix). Much of the original interest in the dimer model arose because it provides a simple and elegant solution for the 2–dimensional Ising model [5].

The problem of enumerating perfect matchings is of course a classical problem in graph theory and combinatorics (see *e.g.* [6]), and can also be phrased in terms of domino tilings [7]. During the last years, the interest in the dimer model was revived thanks to its manifold connections to other branches of mathematics and physics, such as the topological string A–model [8, 9], real algebraic geometry [10, 11], BPS black holes from D–branes wrapping collapsed cycles [12] and supersymmetric quantum mechanics and categorification techniques [13]. Furthermore, a correspondence between the dimer model and quiver gauge theories arising from $D3$–branes probing a singular toric surface was discovered and worked out in great detail [14, 15, 16, 17, 18]. An explanation of this correspondence via mirror symmetry was given in [19].

The plan of this note is as follows. We briefly introduce the dimer model and give a Potts–like description of the dimer model living on a $2 \times n$ strip on the torus. Using this description, we derive a closed formula for the Newton polynomial for any value of n. The same result can also be derived with a recursion relation obtained by mapping the problem to a one–dimensional monomer–dimer system, and is given both for a strip of squares and a strip of hexagons.

Furthermore, the question is translated to the problem of finding a minimal feedback arc set on the dual graph.

A *bipartite* graph $\mathscr{G}$ is a graph in which all vertices can be colored black or white, such that each black vertex is only connected by links to white vertices and vice versa. Let M be a subset of the set E of edges of $\mathscr{G}$. M is called a *matching*, if its elements are links and no two of them are adjacent. If every vertex of $\mathscr{G}$ is saturated under M, the matching is called *perfect*. Such a link that joins a black and a white vertex is called a *dimer*. The *dimer model* describes the statistical mechanics of a system of random perfect matchings. In the simplest case, we ask for the number of close packed dimer configurations, *i.e.* the number of perfect matchings.

Kasteleyn [1, 2] introduced an orientation on $\mathscr{G}$, which leads to a signed adjacency matrix K, now called the *Kasteleyn matrix*. The Pfaffian of K gives the number of perfect matchings. A *Kasteleyn orientation* fulfills the following condition: the product of all edge weights around a face must equal -1 if the number of edges around the face is $0 \mod 4$. If the number of edges equals $2 \mod 4$, the product must equal 1 [6]. One can choose an orientation by consistently assigning arrows to the edges of the graph, as originally suggested by Kasteleyn [1, 2]. The above treatment can be straightforwardly generalized to any genus g Riemann surface.

In the following, we will restrict ourselves to regular $2 \times n$ graphs $\mathcal{G}_{2,n}$ embedded on a torus, to which we will refer in the following as a *strip*. On the torus, there are two non–trivial cycles, which we will denote by z and w.
In the case of the plane graph, the edge weights originated solely from the Kasteleyn orientation. We choose a positive direction on the dimers, say $\bullet \to \circ$. Now we assign the weight z (w) to each edge which crosses the cycle z (w) in positive direction and the weight $1/z$ ($1/w$) to each edge which crosses it in negative direction. While the Pfaffian of the Kasteleyn matrix yielded a number in the case of the plane graph, it becomes a polynomial in z and w on the torus, the so–called characteristic polynomial or *Newton polynomial* of the graph. The coefficient of each monomial $z^p w^q$ gives the number of matchings with *weight* $(z, w) = (p, q)$. These are matchings with the number of dimers crossing z in positive direction minus the number of dimers crossing z in negative direction equal to p (analogous for q). In the literature, what we call the weight is usually referred to as the slope of a height function defined on the composition of two matchings.[1] The matching shown in Figure 1 has weight $(1, 0)$, where $1 = 2 - 1$.

The partition function, or Newton polynomial, of the dimer model on the torus takes the form

$$\mathcal{P}_{m,n}(z, w) = \sqrt{\det K} = \sum_{n_z, n_w} N_{n_z,n_w} (-1)^{n_z+n_w+n_z n_w} z^{n_z} w^{n_w}, \tag{1}$$

where the N_{n_z,n_w} count the number of matchings of weight (height change) (n_z, n_w). Furthermore, the total number of matchings for the square graph on the torus is given by

$$Z_{m,n} = \frac{1}{2} \left(-\mathcal{P}^{\text{sq}}_{m,n}(1, 1) + \mathcal{P}^{\text{sq}}_{m,n}(1, -1) + \mathcal{P}^{\text{sq}}_{m,n}(-1, 1) + \mathcal{P}^{\text{sq}}_{m,n}(-1, -1) \right), \tag{2}$$

[1] The height function is defined as follows. Choose a reference matching PM_0. To find the slope of a matching PM, compose it with the reference matching, $\text{PM} - \text{PM}_0$, where the minus serves to change the orientation of PM_0 to $\circ \to \bullet$. This results in closed loops (composition cycles) and double line dimers. The rule is that when an edge in PM belonging to a closed loop is crossed such that the black node is to its left (right), the height changes by $+1$ (-1). If an edge belonging to PM_0 is crossed, the signs are reversed. This height function is defined up to the choice of the reference matching PM_0. Crossing the boundary of the fundamental region of the torus, this function can jump. If the height function jumps by p units crossing z, it is associated to the power z^p in the Newton polynomial of the graph (and equivalently for w). Choosing a different reference matching results in a common prefactor of $z^{p_0} w^{q_0}$ for all monomials. Our method of assigning weights to a matching corresponds to choosing a reference matching of weight $(0, 0)$ that does not intersect the z or w cycle.

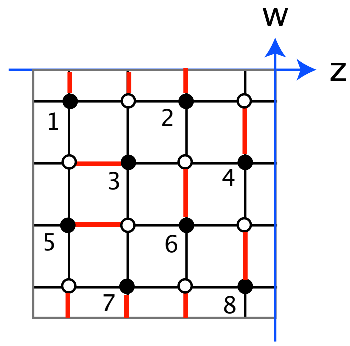


Figure 1: Example of a square graph on the torus

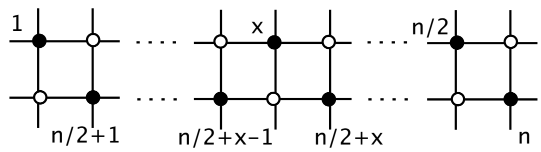


Figure 2: Long strip of squares on the torus

where the first term is always zero.

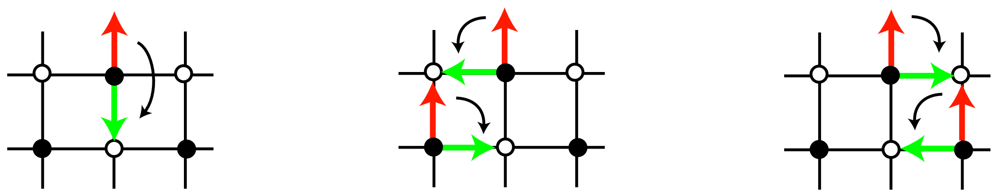
2 Combinatorics for the strip of squares

We consider a long strip of squares $\mathscr{G}_{2,n}$ on the torus, see Figure 2, focusing on the z–weights. A strip containing n black nodes can accommodate matchings with weights $n/2, n/2 - 1, \ldots, -(n/2-1), -n/2$, *i.e.* there are $n+1$ subsets. We would like to find a direct way of obtaining the multiplicities of the matchings of a given weight, *i.e.* the information contained in the characteristic polynomial, with just the number n as input data. Using the Kasteleyn construction, we find the following multiplicities for the first five cases:

n	z^{-5}	z^{-4}	z^{-3}	z^{-2}	z^{-1}	z^{0}	z^{1}	z^{2}	z^{3}	z^{4}	z^{5}
2					1	4	1				
4				1	8	16	8	1			
6			1	12	48	76	48	12	1		
8		1	16	96	272	384	272	96	16	1	
10	1	20	160	660	1520	2004	1520	660	160	20	1

(3)

The above sequences do not have an obvious structure. We will solve the problem using an operator perspective.



(a) $\mathrm{d}_a(x)$ (b) $\mathrm{d}_b(x, n/2 + x - 1)$ (c) $\mathrm{d}_b(x, n/2 + x)$

Figure 3: The three basic operations that can be performed on a spin pointing up

2.1 Potts picture derivation

In order to find a convenient formalism, we use the following picture: we attach to each black node a $\mathbb{Z}_m$ spin, which can point along all the directions in which the black node is joined to a white node by an edge. This results in a description reminiscent of the m–state Potts model, where m here is the valency of the nodes. A hexagon graph results in a 3–state model, while the square graph gives a 4–state model, in which the spin can point up, down, left, or right: $|\uparrow\rangle$, $|\downarrow\rangle$, $|\rightarrow\rangle$, $|\leftarrow\rangle$. To describe a dimer configuration, we take the spins to point to those white nodes which are joined to the black nodes by a dimer. Since we are interested in perfect matchings, we must restrict the possible configurations of the Potts model to those, in which each node is only touched by one dimer.

We label the black nodes by numbers as shown in Figure 1. The perfect matching shown in Figure 1 can be written as the following spin state:

$$|\uparrow\uparrow\leftarrow\uparrow\rightarrow\uparrow\downarrow\uparrow\rangle \ . \tag{4}$$

As we will show in the following, it is possible to define basic operations on the spins allowing us to reach all states starting from the highest weight state.

The highest weight state is unique and in the spin picture is the one with all spins up:

$$|\underbrace{\uparrow\uparrow\uparrow \ldots \uparrow\uparrow}_{n}\rangle =: |n\rangle \ . \tag{5}$$

Also the lowest weight state is unique and is the one with all spins down. We are now looking for basic operations on the spins, which take us from the highest weight state to another state. A spin pointing up can be flipped to three possibilities: down, left and right. Given the structure of the highest weight matching, flipping a spin down gives another perfect matching, while flipping the spin on node x left requires us to flip the spin on node $n/2 + x - 1$ right to arrive at a perfect matching. Flipping the spin on node x right requires us to flip the spin at $n/2 + x$ to the left. See Figure 3 for these three basic operations. Each of

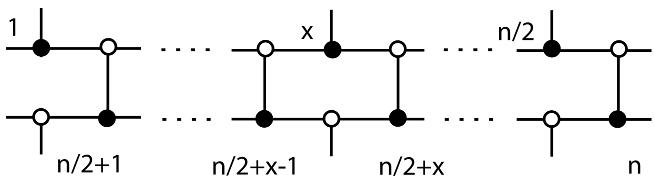


Figure 4: Long strip of hexagons on the torus

these operations lowers the weight of the perfect matching by one. We shall denote the spin flip from up to down at node x by $\mathrm{d}_a(x)$. The spin flips by $\pi/2$ on the nodes x and y will be denoted by $\mathrm{d}_b(x,y)$. For $x = 2, \ldots, n/2$, $y = n/2 + x - 1$ or $y = n/2 + x$ and for $x = 1$, $y = n/2 + x$ or $y = n$.

To familiarize ourselves with this operator picture, see the example of $n = 4$ black nodes in Appendix A.

Before we treat the square graph, we first solve the case of the strip of hexagons, which can be obtained from the square by deleting half of the vertical links, see Figure 4.

2.1.1 Combinatorics for the strip of hexagons

Since the strip of hexagons is a 3–state model and does not accommodate downwards (or upwards, depending on which half of the vertical links we choose to delete) pointing spins, the weights in z only run from 0 to $n/2$ and only the spin flips by $\pi/2$, $\mathrm{d}_b(x,y)$, exist.

The first five cases have the following multiplicities:

n	z^0	z^1	z^2	z^3	z^4	z^5
2	2	1				
4	2	4	1			
6	2	9	6	1		
8	2	16	20	8	1	
10	2	25	50	35	10	1

(6)

The combinatorics therefore stems exclusively from repeated operations of d_b on the state $|n\rangle$. On the hexagon graph, the example of $n = 4$ black nodes shown in the Appendix reduces to what is represented in Figure 5.

Theorem 2.1. *The general formula for the number of perfect matchings with z–weight $n/2 - p$ on the strip of hexagons on the torus is*

$$a_{n,p} = \frac{n}{p!} \prod_{q=1}^{p-1} (n - p - q) = \frac{n}{n-p} \binom{n-p}{p} . \tag{7}$$

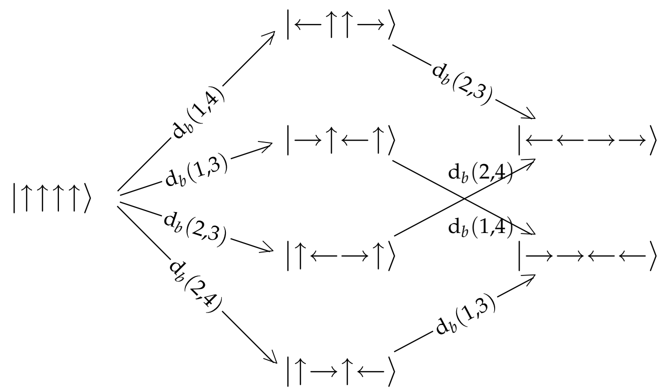


Figure 5: Strip of hexagons with $n = 4$ black nodes

Proof. The operator description explained above is convenient because it allows to map the two–dimensional dimer problem on the hexagonal strip to a one–dimensional monomer–dimer problem. In fact we can simply concentrate on the upper line of the strip and consider two nodes to be occupied by a dimer if they are occupied by a horizontal line in the strip resulting from a b–move, and to be occupied by a monomer if the dimer is vertical in the strip. The occupation of the lower line is uniquely determined by consistency. Let us now write down the partition function for a monomer–dimer system living on a $1D$ lattice of n nodes with periodic boundary conditions (note that we do not require the lattice to be bipartite anymore, so n can be odd). The partition function reads:

$$Q_n(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,p}\, q^p \,, \tag{8}$$

where $a_{n,p}$ is the number of configurations with p dimers. Let us first start with a slightly simpler system of n nodes on a line with free boundary conditions, and let $P_n(q)$ be the corresponding monomer–dimer partition function. This function satisfies the following recursion relation:

$$P_{n+1}(q) = P_n(q) + q\, P_{n-1}(q) \,. \tag{9}$$

This can be understood as follows. When adding an extra point after the n-th, one can either add a monomer which leaves $P_{n+1}(q) = P_n(q)$ (since $P_n(q)$ counts the dimers), or add a dimer (and multiply by q) if the last point was previously occupied by a monomer. The configurations in $P_n(q)$ where the n-th point is a monomer are precisely counted by $P_{n-1}(q)$. Note that this is the q–analogue of the Fibonacci sequence and in fact the actual Fibonacci sequence can be recovered by $\{ P_n(1) \}$, where the $P_n(q)$ are obtained by using the initial

conditions $P_1(q) = 1$, $P_2(q) = 1 + q$.

In a similar way, we can understand the relation between $P_n(q)$ and $Q_n(q)$. In fact, making the plane strip periodic by adding a line between the first and the n-th node can either leave the partition function invariant or add an extra dimer for each configuration where the first and last nodes are occupied by a monomer. Therefore,

$$Q_n(q) = P_n(q) + q\, P_{n-2}(q)\,. \tag{10}$$

It is easy to see that $Q_n(q)$ satisfies the same recursion relation as $P_n(q)$. On the other hand, we are interested in the n even case, so it is better to recast it in the form

$$Q_{n+2}(q) = (1 + 2q)\, Q_n(q) - q^2 Q_{n-2}(q)\,, \tag{11}$$

that can be easily solved with the initial conditions

$$Q_0(q) = 1\,, \qquad\qquad Q_2(q) = 1 + 2\,q\,, \tag{12}$$

the solution being

$$Q_n(q) = \frac{1}{2^n}\left[\left(1 - \sqrt{1+4\,q}\right)^n + \left(1 + \sqrt{1+4\,q}\right)^n\right]\,. \tag{13}$$

To expand (13) in powers of q we can use the identity

$$(a+b)^m + (a-b)^m = 2\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2\,k} a^{m-2k}\, b^{2k}\,, \tag{14}$$

and find

$$Q_n(q) = \frac{2}{2^n}\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2\,k} (1+4\,q)^k = \frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2\,k} \sum_{p=0}^{k} \binom{k}{p} (4\,q)^p\,. \tag{15}$$

Inverting the sums, we find

$$Q_n(q) = \frac{1}{2^{n-1}}\sum_{p=0}^{\lfloor n/2 \rfloor} (4\,q)^p \sum_{k=p}^{\lfloor n/2 \rfloor} \binom{n}{2\,k}\binom{k}{p}\,. \tag{16}$$

Using the identity (see [20])

$$\sum_{k=p}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{k}{p} = 2^{n-1-2p}\frac{n}{n-p}\binom{n-p}{p}, \tag{17}$$

we obtain

$$Q_n(q) = \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{n}{n-p} \binom{n-p}{p} q^p \,. \tag{18}$$

□

The explicit expression for $Q_n(q)$ allows to compactly summarize the result for any strip by defining the generating function

$$\tilde{\mathcal{F}}^{\text{hex}}(s,q) = \sum_{n=0}^{\infty} s^n Q_n(q) = \frac{s-2}{s^2 q + s - 1} \,. \tag{19}$$

Corollary 2.2. *The partition function for the strip of hexagons on the torus, considering only z is:*

$$\begin{aligned} \mathcal{P}_n^{\text{hex}}(z) = z^{n/2} \sum_{p=0}^{n} (-1)^p \, z^{-p} \, a_{n,p} = z^{n/2} Q_n(-1/z) = \\ = \frac{z^{n/2}}{2^n} \left[\left(1 - \sqrt{1 - \frac{4}{z}}\right)^n + \left(1 + \sqrt{1 - \frac{4}{z}}\right)^n \right] \,. \end{aligned} \tag{20}$$

Since there is only one matching with weight w and one with weight $1/w$, the full Newton polynomial on the torus is

$$\mathcal{P}_{2,n}^{\text{hex}}(z,w) = \mathcal{P}_n^{\text{hex}}(z) - w - \frac{1}{w}. \tag{21}$$

Again we can summarize the result into a generating function in z for the dimers on a hexagonal strip as follows:

$$\mathcal{F}^{\text{hex}}(s,z) = \sum_{n=0}^{\infty} \mathcal{P}_n^{\text{hex}}(z) s^n = \frac{2 - \imath s\sqrt{z}}{1 - \imath p\sqrt{z} - s^2} \,. \tag{22}$$

2.1.2 Generalization to the square strip

After having solved the problem for the strip of hexagons, we return to the square strip. We denote a state which only contains up and down spins by $|a\rangle$, while a state which contains at least one left/right pair by $|b\rangle$. Under the action of our two operators, we have

$$\mathrm{d}_a \, |a\rangle = |a\rangle \ , \quad \mathrm{d}_a \, |b\rangle = |b\rangle \ , \quad \mathrm{d}_b \, |a\rangle = |b\rangle \ , \quad \mathrm{d}_b \, |b\rangle = |b\rangle \ . \tag{23}$$

We denote by p the number of times we act on the highest weight state. The resulting states have weight $n/2 - p$. The order in which the state was acted on by the d_a and d_b does not matter, so at level p, there are $p+1$ combinations of d_a and d_b. On the highest weight state $|n\rangle$, we can act in n ways with d_a and in n ways with d_b. On $\mathrm{d}_a \, |n\rangle$, we can then act in $n-1$ ways with d_a and in $n-2$ ways with d_b. On $\mathrm{d}_b \, |n\rangle$, we can act in $n-2$ ways with d_a and in $n-3$ ways by d_b, etc. It is in general easier to compute the number of possibilities of acting

with d_a on a given state than with d_b, so we choose the most convenient path to obtain the full combinatorics, see Figure 6.

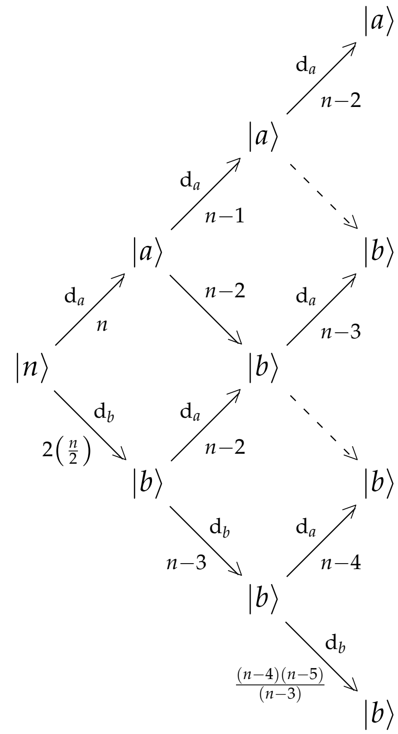


Figure 6: Diagram of d_a, d_b operations

We summarize the results for the first three levels:

$$\begin{array}{ccccccccc}
p & & & & & & & & \\
1 & \begin{array}{c} d_a \\ n \end{array} & + & \begin{array}{c} d_b \\ n \end{array} & & & & & \\
2 & \begin{array}{c} d_a d_a \\ \frac{n(n-1)}{2} \end{array} & + & \begin{array}{c} d_a d_b \\ n\,(n-2) \end{array} & + & \begin{array}{c} d_b d_b \\ \frac{n(n-3)}{2} \end{array} & & & \\
3 & \begin{array}{c} d_a d_a d_a \\ \frac{n(n-1)(n-2)}{3!} \end{array} & + & \begin{array}{c} d_a d_a d_b \\ \frac{n(n-2)(n-3)}{2!} \end{array} & + & \begin{array}{c} d_b d_b d_a \\ \frac{n(n-3)(n-4)}{2!} \end{array} & + & \begin{array}{c} d_b d_b d_b \\ \frac{n(n-4)(n-5)}{3!} \end{array} &
\end{array} \tag{24}$$

Note the denominators, which avoid an overcounting of states obtained by repeatedly applying the same operators. The result for the p-th level is a sum of $p+1$ terms. The denominator of the q-th term on level p is $q!\,(p-q)!, \quad q = 0, .., p$.

Theorem 2.3. *The q-th term on level p equals to*

$$b_{n,p,q} = \frac{1}{q!(p-q)!}\, n\, \frac{(n-q-1)!}{(n-q-p)!} = \frac{n}{n-q}\binom{p}{q}\binom{n-q}{p}, \quad q = 0, \dots, p\,. \tag{25}$$

The total number of perfect matchings with weight $n/2 - p$ is therefore

$$b_{n,p} = \sum_{q=0}^{p} \frac{n(n-q-1)!}{q!(p-q)!(n-q-p)!}\,. \tag{26}$$

Note that $b_{n,n-p} = b_{n,p}$, which mirrors the symmetry of the sequence.

Proof. We will prove (25) by induction. The first three terms are shown explicitly in (24). Note that $(p-q)$ counts the number of times d_a has been applied and q counts the number of times d_b has been applied. One can act with d_a on a given configuration at level (p,q) in $n-q-p$ ways, since of the n upwards pointing arrows in the highest weight state, $(p-q)$ were turned down by acting with d_a, whereas $2\,q$ were turned horizontally by acting q times with d_b. This number has to be normalized by the degeneracy factor $q!(p-q)!$. Indeed,

$$q!(p+1-q)!\, b_{n,p+1,q} = (n-p-q)\, q!\,(p-q)!\, b_{n,p,q}\,. \tag{27}$$

Since every possible state can be arrived at by acting with d_a on a state created by acting with only d_b on $|n\rangle$, it is enough to now prove (25) for the case $p = q$, which corresponds to the case of the strip of hexagons. In fact $b_{n,p,p} = a_{n,p}$ which was calculated in the last section. □

Corollary 2.4. *The Newton polynomial in z for the strip with n black nodes (n even) is*

$$\mathcal{P}^{\mathrm{sq}}_n(z) = z^{-n/2} \sum_{p=0}^{n} (-1)^p z^p\, b_{n,p} = z^{-n/2} \sum_{p=0}^{n} (-1)^p z^p \sum_{q=0}^{p} \frac{n(n-q-1)!}{q!(p-q)!(n-q-p)!}. \tag{28}$$

This constitutes a compact sum formula for the Newton polynomial for the strip of squares of arbitrary length. Since there is only one matching with weight w and one with weight $1/w$, the full Newton polynomial on the torus is

$$\mathcal{P}^{\mathrm{sq}}_{2,n}(z,w) = \mathcal{P}^{\mathrm{sq}}_n(z) - w - \frac{1}{w}. \tag{29}$$

2.1.3 Recursion relation

Even though we already have the result for the partition function in the form of a sum, we will now derive a recursion relation which expresses $\mathcal{P}^{\mathrm{sq}}_n(z)$ through $\mathcal{P}^{\mathrm{sq}}_{n-2}(z)$, *i.e.* we take a strip $\mathcal{G}_{2,n-2}$ and add in another square, resulting in $\mathcal{G}_{2,n}$. Like in the case of the hexagon strip, this will provide us with a closed form, and like for the hexagon, the derivation invokes the map to a monomer–dimer system.

Theorem 2.5. *The recursion relation for the partition function of the strip of squares on the torus is*

$$\mathcal{P}_n^{\rm sq}(z) = \mathcal{P}_2^{\rm sq}(z)\,\mathcal{P}_{n-2}^{\rm sq}(z) - \mathcal{P}_{n-4}^{\rm sq}(z) \tag{30}$$

with the initial conditions

$$\mathcal{P}_0^{\rm sq}(z) = 1\,, \qquad\qquad \mathcal{P}_2^{\rm sq}(z) = 4 - z - \frac{1}{z}\,. \tag{31}$$

The solution to (30) is given by

$$\mathcal{P}_n^{\rm sq}(z) = \frac{1}{(-4\,z)^{n/2}}\left[\left(z-1-\sqrt{1+z\,(z-6)}\right)^n + \left(z-1+\sqrt{1+z\,(z-6)}\right)^n\right] \tag{32}$$

This result is equivalent to (28) and provides a closed form.

Proof. The dimer model on the square strip can be mapped to a one–dimensional monomer–dimer in the following way (see Fig. 7). A dimer living on a z link corresponds to a u–monomer living on the $\bullet$ node, a dimer living on a $1/z$ link corresponds to a v–monomer on the $\circ$ node, a dimer living on an internal vertical link corresponds to an empty point, and a horizontal dimer is again a dimer. In this way, the square strip problem becomes a monomer^3–dimer on a one–dimensional n–node lattice. As for the case of the hexagon graph, let us start with free boundary conditions described by the partition function

$$P_n(u,v,t) = \sum_{i,j,k} c_{i,j,k} u^i v^j t^k\,, \tag{33}$$

where $c_{i,j,k}$ is the number of configurations with i u–monomers, j v–monomers, k dimers and $(n-i-j-2k)$ free nodes.

We can always suppose without loss of generality that the first node is a $\bullet$, such that $P_1(u,v,t) = 1+u$. Adding a $\bullet$ node to a string of $2\,k$ points results in

$$P_{2\,k+1}(u,v,t) = (1+u)\,P_{2\,k}(u,v,t) + t\,P_{2\,k-1}(u,v,t)\,, \tag{34}$$

which can be read as stating that the new point is either free, occupied by a u–monomer or, if the $2\,k$-th node was free, by a new dimer. Similarly, adding a $\circ$ node to a string of $2\,k+1$ gives

$$P_{2\,k+2}(u,v,t) = (1+v)\,P_{2\,k+1}(u,v,t) + t\,P_{2\,k}(u,v,t)\,. \tag{35}$$

Adding periodic boundary conditions means adding a new line between nodes 1 and n. One can see that the corresponding partition function is given by

$$Q_n(u,v,t) = P_n(u,v,t) + t\,P_{n-2}(u,v,t)\,, \tag{36}$$

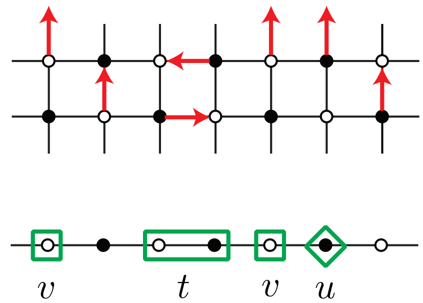


Figure 7: Mapping of a square strip configuration to a monomer–dimer configuration. This particular configuration contributes as uv^2t to the partition function.

which can be understood as saying that we get either the same configurations or a new dimer if the extremal nodes were both empty. Since the relation between Q and P is linear, they satisfy the same recurrence equation. On the other hand, Q_n is well defined only for n even, so the equation is better cast into the form

$$Q_{n+2}(u,v,t) = [(1+u)(1+v)+2t]\, Q_n(u,v,t) - t^2 Q_{n-2}(u,v,t)\,, \tag{37}$$

and solved with the initial conditions

$$Q_0(u,v,t) = 1\,, \qquad\qquad Q_2(u,v,t) = 1+u+v+uv+t\,. \tag{38}$$

The partition function for the dimer on the square strip is then obtained using the weights as they were defined on the initial bipartite graph and reads

$$\mathcal{P}_n^{\mathrm{sq}}(z) = Q_n(-z,-1/z,1)\,. \tag{39}$$

Hence it satisfies

$$\mathcal{P}_{n+2}^{\mathrm{sq}}(z) = \left(4 - z - \frac{1}{z}\right)\mathcal{P}_n^{\mathrm{sq}}(z) - \mathcal{P}_{n-2}^{\mathrm{sq}}(z)\,, \tag{40}$$

with the initial conditions (31). Solved explicitly, the closed form (32) is obtained. □

The expression in (32) can be summarized by introducing a generating function in z as follows.

$$\mathcal{F}^{\mathrm{sq}}(s,z) = \sum_{n=0}^{\infty} \mathcal{P}_n^{\mathrm{sq}}(z) s^n = \frac{4\imath s\,(z-1)\sqrt{z} + 8z}{4\imath s\,(z-1)\sqrt{z} + 4z - 4s^2 z}\,. \tag{41}$$

2.2 Comparison with Kasteleyn's product formula

For the total number of matchings on an $m \times n$ square lattice on the torus, a product formula exists [2]:

$$Z_{m,n} = -\frac{1}{2} \sum_{a,b=0}^{1} (-1)^{a+b+ab} \prod_{k=1}^{m/2} \prod_{l=1}^{n} 2 \sqrt{x^2 \sin^2 \frac{(2\,k-a)\,\pi}{m} + x'^2 \sin^2 \frac{(2\,l-b)\,\pi}{n}}\,. \tag{42}$$

Here, x, x' are the edge weights which in our case are equal to one. When specializing formula (42) to $m = 2$, this gives

$$\begin{aligned} Z_{2,n} &= -\frac{1}{2} \prod_{l=1}^{n} 2 \left| \sin \frac{2\pi l}{n} \right| + \frac{1}{2} \prod_{l=1}^{n} 2 \left| \sin \frac{(2l-1)\pi}{n} \right| + \\ &\quad + \frac{1}{2} \prod_{l=1}^{n} 2 \sqrt{1 + \sin^2 \frac{2\pi l}{n}} + \frac{1}{2} \prod_{l=1}^{n} 2 \sqrt{1 + \sin^2 \frac{(2l-1)\pi}{n}} = \\ &=: -A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{43}$$

Note, that the first term, A_1, is identical to zero (already for general m). Furthermore, $A_2 = 2$, and

$$A_2 - A_1 = 2, \quad A_4 - A_3 = 2 \quad \forall\, n. \tag{44}$$

From the product formula (43), this is little obvious, but clear from the point of view of formula (2) and the Newton polynomial $\mathcal{P}^{\text{sq}}_{2,n}$. Since the monomials in w are $-w$ and $-1/w$, $\mathcal{P}^{\text{sq}}_{2,n}(\,\cdot\,,1)$ and $\mathcal{P}^{\text{sq}}_{2,n}(\,\cdot\,,-1)$ must differ by four. Therefore, we arrive at the identifications

$$A_1 = \tfrac{1}{2}\,\mathcal{P}^{\text{sq}}_{2,n}(1,1), \quad A_2 = \tfrac{1}{2}\,\mathcal{P}^{\text{sq}}_{2,n}(1,-1), \quad A_3 = \tfrac{1}{2}\,\mathcal{P}^{\text{sq}}_{2,n}(-1,1), \quad A_4 = \tfrac{1}{2}\,\mathcal{P}^{\text{sq}}_{2,n}(-1,-1). \tag{45}$$

Substituting (45) into (28), this means at the same time that

$$\mathcal{P}^{\text{sq}}_{n}(1) = \sum_{p=0}^{n} (-1)^p \sum_{q=0}^{p} \frac{n(n-q-1)!}{q!(p-q)!(n-q-p)!} = 2. \tag{46}$$

2.3 General case

We now consider the general case of a square lattice on the torus with $n \times m$ nodes ($n/2 \times m/2$ black nodes). In general, the states with a given weight have here a much bigger degeneracy, since the interior of the graph allows for many different configurations which do not affect the boundaries. Also here, the highest weight state is unique and in the spin picture is the one with all spins up. For large examples, it might seem at first surprising that the boundary conditions are strong enough to completely fix the configuration in the interior, but it is easy to convince oneself by inspecting a small example that the interlacing structure of the spins does not allow for any other configuration in the interior. This changes once

one of the spins on the boundary points down. For the next simplest case of $m = 4$ for example, each boundary state with weight $n_{max} - 1$ has a degeneracy of 3 in the interior. It is obvious that this degeneracy grows quickly with growing m and lower weight. Contrary to the case before, one must always change more than one spin to arrive at a new matching. On the whole we see that the spin picture is not very well adapted to the general case and the complexity of the derivation used for the strip rises to an unreasonable level for $p > 1$. Also ansätze for recursion relations have turned out to be of little use. To solve the general problem, other methods might be more appropriate.

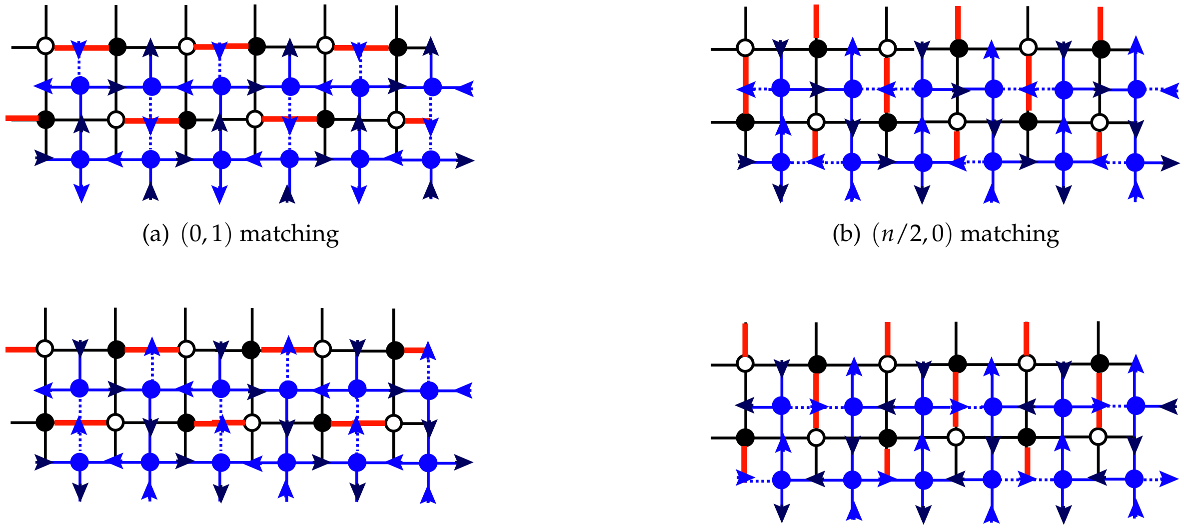
3 Translation to the minimal feedback arc set problem

The problem of finding a minimal set of arcs in a directed graph upon the deletion of which the graph becomes acyclic is well studied in mathematics and computer science [21]. Such a set is called a minimal *feedback arc set* (FAS). There exists a precise relation between the problem of finding all minimal FAS of a digraph and the problem of identifying all the perfect matchings in its dual graph.

Consider a bipartite plane graph $\mathscr{G}$ with N nodes, and its graph dual $\mathscr{G}'$ (where nodes become faces, faces become nodes and edges remain edges) . The dual graph $\mathscr{G}'$ becomes a digraph if the edges around a face corresponding to a black node in $\mathscr{G}$ are oriented clockwise, while the edges circling a face corresponding to a white node are oriented counterclockwise. The one–cycles in $\mathscr{G}'$ are generated by the plaquettes $\{ p_j \}_{j=1}^{N}$ which correspond to the vertices of $\mathscr{G}$. Removing the edge e_{ij} shared by the cycles p_i and p_j breaks both cycles. A minimal FAS is obtained taking the collection of $N/2$ edges $\{ e_{i_k j_k} \}_{k=1}^{N/2}$ shared by disjoint pairs of plaquettes p_{i_j} and p_{i_k}. In the dual graph $\mathscr{G}$, this corresponds to selecting a set of edges joining all the nodes and touching all of them only once. In other words, a minimal FAS in $\mathscr{G}'$ is a perfect matching in $\mathscr{G}$.

The situation is different when $\mathscr{G}$ is embedded on a Riemann surface of genus $g > 0$, because in addition to the one–cycles generated by the plaquettes, there are $2g$ equivalence classes of cycles of non–trivial holonomy. In this case, the winding cycles in $\mathscr{G}'$ are generated by the zig–zag paths[2]. It follows that being a perfect matching in $\mathscr{G}$ is only a necessary condition for a set of edges to be a FAS in $\mathscr{G}'$. Let us again restrict ourselves to the case of $g = 1$. A useful way to represent the partition function in Eq. (1) consists in drawing a point in the (z, w) plane at coordinates (n_z, n_w) for each monomial of the form $z^{n_z} w^{n_w}$, corresponding to the matchings with weight (n_z, n_w). In this way, one obtains the region of a $\mathbb{Z}^2$ lattice delimited by a convex polygon, the *Newton polygon*. We can hence distinguish between internal and boundary points (or matchings). It was shown in [18] that a perfect matching corresponding to an internal point of this Newton polygon is always a FAS. Removing a

[2] A zig–zag path is a path which turns alternatingly maximally left and maximally right at the vertices.



(a) $(0,1)$ matching

(b) $(n/2,0)$ matching

(c) $(0,-1)$ matching

(d) $(-n/2,0)$ matching

Figure 8: Feedback arc sets on the dual graph to the square strip. The dual graph is represented in light blue. The removal of the boundary matchings (dashed lines) preserves a zig–zag path (dark blue).

boundary matching, on the other hand, always preserves at least one zig–zag path.

In the case at hand, *i.e.* for the square strip, the polygon is a rhombus with the four corner vertices at the points $(\pm n/2, 0)$ and $(0, \pm 1)$. This means that there are four boundary matchings that will not be feedback arc sets. In fact, removing the arrows corresponding to these matchings will always preserve exactly one zig–zag path (see Fig. 8).

Theorem 3.1. *On a $2 \times n$ digraph on the torus (n even), there are*

$$N_n^{\mathrm{FAS}} = \left(-1+\sqrt{2}\right)^n + \left(-1-\sqrt{2}\right)^n - 2 \tag{47}$$

minimal feedback arc sets. This number is generated by the function

$$\mathcal{F}^{\mathrm{FAS}}(s) = \sum_{n=0}^{\infty} N_n^{\mathrm{FAS}} s^n = \frac{2\left(1+\sqrt{2}\right)+\left(\sqrt{2}-4\right)s}{(s-1)\left(-1+\left(-1+\sqrt{2}\right)s\right)} = \mathcal{F}^{sq}(s,-1) - \frac{2}{1-s}\,. \tag{48}$$

Proof. Follows from combining Eq. (2) and Eq. (32) and subtracting the four boundary matchings. □

4 Discussion and further directions

In this note, we have derived an explicit formula for the partition function of the dimer model on a $2 \times n$ strip of squares or hexagons on the torus. This result was derived via an operator picture, and by mapping the problem to a one–dimensional monomer–dimer system. This result was furthermore translated to the question of finding the minimal feedback arc sets on the dual graph.

An obvious continuation of this work would be the generalization to an $n \times m$ graph. This problem turns out to be much more complicated than the one addressed here and the methods employed here have proven not to be properly adapted to the more general question.

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A Example: $n = 4$ square strip

To familiarize ourselves with this operator picture, we consider the example of $n = 4$ black nodes, see Figure 9. We start with the highest weight state, which has has weight 2. The solid arrows in the diagram denote the action by d_a, the dotted arrows the action by d_b. We find a symmetric sequence structure.

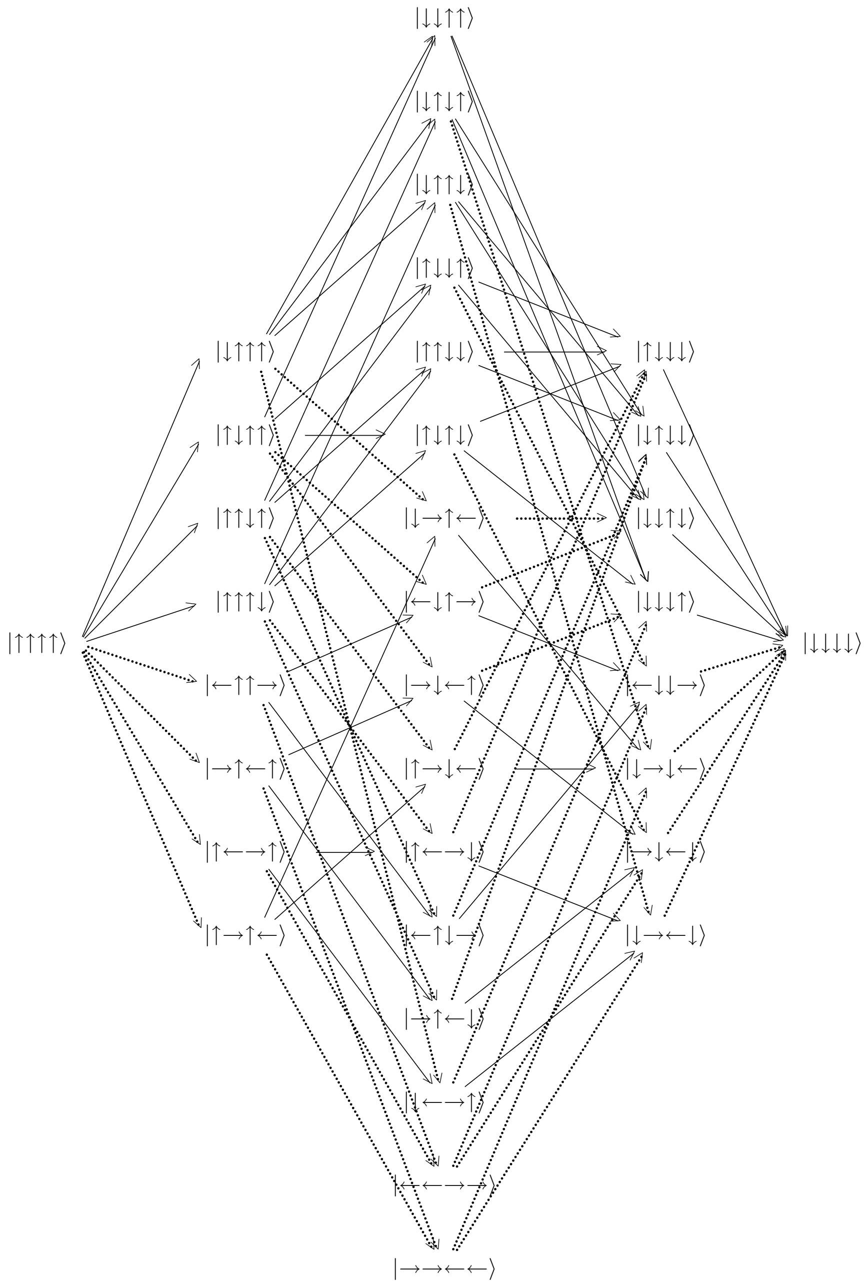

Figure 9: Strip with $n = 4$ black nodes

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